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Parallelepipeds in Sets of Integers

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A *d-dimensional parallelepiped* in $\mathbb{N}$ is a set of the form $\{m+\sum_{i \in S} m_i: S \subseteq \{1, 2,..., d\}\}$ for some positive integers $m, m_1, m_2,..., m_d$. It is proved that a subset of $\{1, 2,..., N\}$ not containing a d-dimensional parallelepiped is of cardinality not exceeding $N^{1-1/2^{d-1}}+O(N^{3/4-1/2^{d-1}})$. A result of a similar nature is established for parallelepipeds satisfying $m_1 | m_2 | \cdots | m_d$. © 1987 Academic Press, Inc.

1. Introduction and the Main Theorems

Given sets $A_1, A_2,..., A_k \subseteq \mathbb{Z}$ denote

$$\sum_{i=1}^{k} A_i = \{a_1+a_2+\cdots+a_k; a_i \in A_i, 1 \leqslant i \leqslant k\}.$$

DEFINITION 1.1. A set $P \subseteq \mathbb{N}$ is a *d-dimensional parallelepiped* if it is of the form

$$P = m + \sum_{i=1}^{d} \{0, m_i\}$$

for some $m, m_1, m_2,..., m_d \in \mathbb{N}$.

DEFINITION 1.2. A parallelepiped $P = m + \sum_{i=1}^{d} \{0, m_i\}$ is *arithmetic* if $m_1 | m_2 | \cdots | m_d$.

EXAMPLE 1.1. An arithmetic progression of length $d+1$ forms a d-dimensional arithmetic parallelepiped.

EXAMPLE 1.2. The set of all integers between 10^{d-1} and 10^d, all of whose digits are either 4 or 7, forms a d-dimensional arithmetic parallelepiped.

For any positive integers d and N, denote by $P_d(N)$ the maximal

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cardinality of a subset of $\{1, 2, \ldots, N\}$ not containing a d-dimensional parallelepiped, and by $A_d(N)$ the corresponding number for arithmetic parallelepipeds. The purpose of this paper is to obtain upper bounds for $P_d(N)$ and $A_d(N)$.

The first result in this direction is due to Hilbert, who proved that, given any finite colouring of $\mathbb{N}$, for every positive integer d there exists a d-dimensional parallelepiped having infinitely many monochromatic translates (for this and related results see, for example, [6].) In [3, Lemma 5.2], as a means for proving a certain pointwise ergodic theorem for endomorphisms of compact abelian groups, it is proved that for any $\varepsilon > 0$ we have

$$P_d(N) < (2+\varepsilon)\, N^{1-1/2^{d-1}} \tag{1.1}$$

for all sufficiently large N. An additional source of interest in the behaviour of $P_d(N)$ and $A_d(N)$ stems from Szemerédi's theorem on the existence of arbitrarily long arithmetic progressions in sets of integers of positive density [9]. For, besides the fact that here we study the existence in $\{1, 2, \ldots, N\}$ of configurations more general than arithmetic progressions, parallelepipeds actually play an important role in the original proof of Szemerédi's theorem. The following improvement of (1.1) will be obtained here.

THEOREM 1.1. *For every positive integer d*

$$P_d(N) < N^{1-1/2^{d-1}} + O(N^{3/4-1/2^{d-1}}).$$

The theorem generalizes a result of Erdös and Turán [5] related to Sidon sets. In fact, they proved that if $1 \leqslant a_1 < a_2 < \cdots < a_r \leqslant N$, with all the sums $a_i + a_j$, $i \leqslant j$, being different, then $r < N^{1/2} + O(N^{1/4})$. This is clearly equivalent to the assertion of Theorem 1.1 in the special case $d = 2$. It is worthwhile to remark that the correct magnitude of the error term here is as yet unknown. An "old conjecture" of Erdös and Turán (cf. [5, p. 55]) states that $r < N^{1/2} + O(1)$, namely that the $O(N^{1/4})$ can be reduced to $O(1)$. However, the main term cannot be improved in this case; a theorem of Bose and Chowla implies that $P_2(N) > N^{1/2} - O(N^{5/16})$ (for an exhaustive treatment of these results see Halberstam and Roth [7]). It would be interesting to obtain lower bounds for $P_d(N)$ for general d. (The same applies to $A_d(N)$ in Theorem 1.2 *infra.*)

Tor present the corresponding result for arithmetic parallelepipeds we define a sequence $(\beta_d)_{d=1}^{\infty}$ by:

$$\beta_d = \frac{\prod_{i=1}^{d} (2^i - 1)^{2^{i-1}/(2^d-1)}}{2^{d-1-(d-1)(d-2)/2(2^d-1)}}, \qquad d = 1, 2, \ldots.$$

THEOREM 1.2. *For every positive integer d*

$$A_d(N) < \beta_d N^{1-1/(2^d-1)} + O(N^{1-2/(2^d-1)}).$$

The general behaviour of the sequence (β_d) is described in

LEMMA 1.1. *(β_d) has the following properties*:

(i) $\beta_{d+1} = \beta_d^{(2^d-1)/(2^{d+1}-1)}((2^{d+1}-1)/2^{d+1-d/2^d})^{2^d/(2^{d+1}-1)}$, $d=1,2,\ldots$;

(ii) $\beta_d > 1$, $d = 2, 3, \ldots$;

(iii) (β_d) *is eventually decreasing*;

(iv) $\lim_{d\to\infty} \beta_d = 1$.

The proof is routine.

We note that much larger subsets of $\{1, 2,\ldots, N\}$ have to be taken to ensure that they contain arithmetic progressions. In fact, it was shown ([1,2, 8]) that, given any $\varepsilon > 0$, one can find, for arbitrarily large N, subsets of $\{1, 2,\ldots, N\}$ of cardinality exceeding $N^{1-\varepsilon}$ containing no arithmetic progression of length three.

One can define in an obvious manner infinite-dimensional parallelepipeds and infinite-dimensional arithmetic parallelepipeds. It is interesting to note that any infinite-dimensional parallelepiped contains an infinite-dimensional arithmetic parallelepiped. Thus, for example, in Hindman's theorem (cf. [6]), which states that if $\mathbb{N} = \bigcup_{i=1}^r A_i$ then for some i there exists a sequence $(a_n)_{n=1}^\infty$ such that all the a_n's and all sums of finitely many distinct a_n's belong to A_i, we can add the requirement that $a_n \mid a_{n+1}$ for each n.

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2. PROOF OF THEOREM 1.1

The proof is carried out by induction on d.

The case $d=1$ is trivial.

Assume that

$$P_d(N) < N^{1-1/2^{d-1}} + cN^{3/4-1/2^{d-1}}$$

for a certain constant c. We have to show that a similar inequality holds with d replaced by $d+1$.

Suppose $1 \leqslant a_1 < a_2 < \cdots < a_r \leqslant N$, where $\{a_1, a_2,\ldots, a_r\}$ contains no

$(d+1)$-dimensional parallelepiped. Let $k<r$ be an arbitrary fixed positive integer, to be determined later. We have

$$\sum_{0<j-i\leqslant k}(a_j-a_i)=(a_r-a_1)+(a_r+a_{r-1}-a_2-a_1)+\cdots$$
$$+(a_r+a_{r-1}+\cdots+a_{r-k+1}-a_k-a_{k-1}-\cdots-a_1)$$
$$\leqslant N+2N+\cdots+kN\leqslant \tfrac{1}{2}(k+1)^2 N. \tag{2.1}$$

The number of summands on the left-hand side is

$$(r-1)+(r-2)+\cdots+(r-k)=kr-\tfrac{1}{2}k(k+1).$$

Write

$$kr-\tfrac{1}{2}k(k+1)=t(N^{1-1/2^{d-1}}+cN^{3/4-1/2^{d-1}}) \tag{2.2}$$

for a suitable $t\in\mathbb{R}$, and let t_1 and t_2 be the integral and the fractional parts of t, respectively. From the induction hypothesis it follows that for any positive integer l the number of terms equal to l on the left-hand side of (2.1) is less than $N^{1-1/2^{d-1}}+cN^{3/4-1/2^{d-1}}$, which implies that

$$\sum_{0<j-i\leqslant k}(a_j-a_i)\geqslant(1+2+\cdots+t_1+(t_1+1)\,t_2)(N^{1-1/2^{d-1}}+cN^{3/4-1/2^{d-1}})$$
$$\geqslant\tfrac{1}{2}t(t+1)\,N^{1-1/2^{d-1}}. \tag{2.3}$$

By (2.1) and (2.3) we have

$$\tfrac{1}{2}t^2N^{1-1/2^{d-1}}\leqslant\tfrac{1}{2}(k+1)^2 N$$

and therefore

$$t\leqslant(k+1)\,N^{1/2^d}.$$

Now (2.2) yields

$$r\leqslant\frac{1}{2}(k+1)+\frac{t}{k}(N^{1-1/2^{d-1}}+cN^{3/4-1/2^{d-1}})$$
$$\leqslant k+(1+1/k)\,N^{1/2^d}(N^{1-1/2^{d-1}}+cN^{3/4-1/2^{d-1}})$$
$$\leqslant k+N^{1-1/2^d}+\frac{1}{k}N^{1-1/2^d}+2cN^{3/4-1/2^d}$$

Taking $k=[N^{1/4}]+1$ we obtain

$$r<N^{1-1/2^d}+(2c+2)N^{3/4-1/2^d}$$

and thus

$$P_{d+1}(N) < N^{1-1/2^d} + (2c+2)\, N^{3/4-1/2^d}.$$

This proves the theorem.

3. Proof of Theorem 1.2

We proceed here also by induction on d. Only the induction step has to be performed. Assume that

$$A_d(N) < \beta_d N^{1-1/(2^d-1)} + c_1 N^{1-2/(2^d-1)}$$

(here and later in the course of the proof $c_1, c_2, \dots$ denote suitable positive constants.) We want to prove a similar inequality with $d+1$ instead of d. N may be assumed to be arbitrarily large. It will be convenient to assume that $d \geqslant 2$. Actually, when carrying out the ideas used in the general case to prove the theorem for $d=2$ we get a somewhat better result as follows.

Proposition 3.1. $A_2(N) \leqslant \beta_2 N^{2/3}$.

Remark 3.1. It seems to be possible, using our method, to reduce the error term in the theorem for general d as well.

Denote $\eta_d = 1/(2^d-1)$. We shall write β and η for β_d and η_d, respectively. Suppose $1 \leqslant a_1 < a_2 < \cdots < a_r \leqslant N$, where $B = \{a_1, a_2, \dots, a_r\}$ contains no $(d+1)$-dimensional arithmetic parallelepiped. For $1 \leqslant i \leqslant j \leqslant N$ denote by $N_{j,i}$ the number of elements of $\{1, 2, \dots, N\}$ which are congruent to i modulo j. Letting e_j be the least non-negative residue of N modulo j, we get

$$N_{j,i} = \begin{cases} 1 + (N-e_j)/j, & 1 \leqslant i \leqslant e_j \\ (N-e_j)/j, & e_j < i \leqslant j. \end{cases}$$

The number of elements of $B \cap (B-j)$, congruent to i modulo j, is at most $A_d(N_{j,i})$, and so, by the induction hypothesis, is less than $\beta N_{j,i}^{1-\eta} + c_1 N_{j,i}^{1-2\eta}$. Let $k < r$ be an arbitrary fixed positive integer, depending only on d. Write

$$kr - \frac{1}{2}k(k+1) = \sum_{j=1}^{l} \sum_{i=1}^{j} (\beta N_{j,i}^{1-\eta} + c_1 N_{j,i}^{1-2\eta}) + t \tag{3.1}$$

where l is a non-negative integer and

$$0 \leqslant t < \sum_{i=1}^{l+1} (\beta N_{l+1,i}^{1-\eta} + c_1 N_{l+1,i}^{1-2\eta}).$$

By (3.1) we have

$$kN \geqslant \sum_{j=1}^{l} \sum_{i=1}^{j} 1 = \tfrac{1}{2} l(l+1)$$

and so

$$l \leqslant c_2 N^{1/2} \tag{3.2}$$

(where c_2 depends on k.) Since functions of the form $f(x) = x^\alpha$ are concave for $0 < \alpha < 1$ we get

$$\begin{aligned}
kr - \frac{1}{2}k(k+1) &\leqslant \sum_{j=1}^{l+1} \left(e_j \beta \left(\frac{N-e_j}{j} + 1 \right)^{1-\eta} + e_j c_1 \left(\frac{N-e_j}{j} + 1 \right)^{1-2\eta} \right. \\
&\quad \left. + (j-e_j)\, \beta \left(\frac{N-e_j}{j} \right)^{1-\eta} + (j-e_j)\, c_1 \left(\frac{N-e_j}{j} \right)^{1-2\eta} \right) \\
&\leqslant \sum_{j=1}^{l+1} \left(j\beta \left(\frac{e_j}{j} \left(\frac{N-e_j}{j} + 1 \right) + \frac{j-e_j}{j} \cdot \frac{N-e_j}{j} \right)^{1-\eta} \right. \\
&\quad \left. + jc_1 \left(\frac{e_j}{j} \left(\frac{N-e_j}{j} + 1 \right) + \frac{j-e_j}{j} \cdot \frac{N-e_j}{j} \right)^{1-2\eta} \right) \\
&= \beta N^{1-\eta} \sum_{j=1}^{l+1} j^\eta + c_1 N^{1-2\eta} \sum_{j=1}^{l+1} j^{2\eta} \\
&\leqslant \beta N^{1-\eta} \int_1^{l+2} x^\eta \, dx + c_1 N^{1-2\eta} \int_1^{l+2} x^{2\eta} \, dx \\
&\leqslant \beta (1 - 1/2^d)\, N^{1-\eta} (l+2)^{1+\eta} + c_3 N^{1-2\eta} (l+2)^{1+2\eta} \\
&\leqslant \beta (1 - 1/2^d)\, N^{1-\eta} l^{1+\eta} (1 + 4/l) + c_3 N^{1-2\eta} l^{1+2\eta} (1 + 4/l) \\
&\leqslant \beta (1 - 1/2^d)\, N^{1-\eta} l^{1+\eta} + c_4 N^{1-\eta} l^\eta + c_4 N^{1-2\eta} l^{1+2\eta}.
\end{aligned}$$

Therefore

$$r \leqslant \frac{1}{k} \beta (1 - 1/2^d)\, N^{1-\eta} l^{1+\eta} + c_5 N^{1-\eta} l^\eta + c_5 N^{1-2\eta} l^{1+2\eta}. \tag{3.3}$$

Considering the sum $\sum_{0 < j-i \leqslant k} (a_j - a_i)$ as in the proof of Theorem 1.1, we obtain

$$\begin{aligned}
\frac{1}{2} k(k+1)\, N - 1 &\geqslant \sum_{j=1}^{l} \sum_{i=1}^{j} j(\beta N_{j,i}^{1-\eta} + c_1 N_{j,i}^{1-2\eta}) + (l+1)\, t \\
&\geqslant \beta \sum_{j=1}^{l} j \left(e_j \left(\frac{N-e_j}{j} + 1 \right)^{1-\eta} + (j-e_j) \left(\frac{N-e_j}{j} \right)^{1-\eta} \right)
\end{aligned}$$

$$= \beta N^{1-\eta} \sum_{j=1}^{l} j^\eta \left(e_j \left(1 + \frac{j-e_j}{N}\right)^{1-\eta} + (j-e_j)\left(1 - \frac{e_j}{N}\right)^{1-\eta} \right)$$

$$\geqslant \beta N^{1-\eta} \sum_{j=1}^{l} j^\eta \left(e_j \left(1 + (1-\eta)\frac{j-e_j}{N} - c_6 \left(\frac{j-e_j}{N}\right)^2\right) + (j-e_j)\left(1 - (1-\eta)\frac{e_j}{N} - c_6\left(\frac{e_j}{N}\right)^2\right)\right)$$

$$= \beta N^{1-\eta} \sum_{j=1}^{l} j^\eta (j - c_6(e_j(j-e_j)^2 + (j-e_j)\, e_j^2)/N^2)$$

$$\geqslant \beta N^{1-\eta} \sum_{j=1}^{l} j^{1+\eta}(1 - c_6 j^2/N^2)$$

$$\geqslant \beta N^{1-\eta} \int_0^l x^{1+\eta}\, dx - c_7 N^{-1-\eta} \int_1^{l+1} x^{3+\eta}\, dx$$

$$\geqslant \frac{\beta}{2+\eta} N^{1-\eta} l^{2+\eta} - c_8 N^{-1-\eta} l^{4+\eta}. \tag{3.4}$$

Using (3.2) we see that

$$\tfrac{1}{2}k(k+1)\, N \geqslant c_9 N^{1-\eta} l^{2+\eta}$$

and hence

$$l \leqslant c_{10} N^{\eta/(2+\eta)}.$$

It follows that the second term on the right-hand side of (3.4) approaches 0 as $N \to \infty$, whence the latter inequality implies

$$\frac{1}{2}k(k+1)\, N \geqslant \frac{\beta}{2+\eta} N^{1-\eta} l^{2+\eta}$$

so that

$$l \leqslant \left(\frac{(2+\eta)\, k(k+1)}{2\beta}\right)^{1/(2+\eta)} N^{\eta/(2+\eta)}.$$

Now (3.3) yields

$$r \leqslant \frac{1}{k} \beta(1 - 1/2^d)\, N^{1-\eta} \left(\frac{(2+\eta)\, k(k+1)}{2\beta}\right)^{(1+\eta)/(2+\eta)} N^{\eta(1+\eta)/(2+\eta)} + c_{11} N^{1-\eta+\eta^2/(2+\eta)}.$$

Taking $k = 2^{d-1}$ (which can be shown to be the choice producing the best result) and employing Lemma 1.1 we get

$$r \leqslant \frac{\beta}{2^d}\left(\frac{(2^{d+1}-1)\,2^d}{2\beta}\right)^{2^d/(2^{d+1}-1)} N^{1-1/(2^{d+1}-1)} + c_{11} N^{1-2/(2^{d+1}-1)}$$

$$= \beta^{(2^d-1)/(2^{d+1}-1)} (2^{d+1}-1)^{2^d/(2^{d+1}-1)}\, 2^{-d+(d-1)2^d/(2^{d+1}-1)} N^{1-1/(2^{d+1}-1)}$$

$$+ c_{11} N^{1-2/(2^{d+1}-1)} = \beta_{d+1} N^{1-1/(2^{d+1}-1)} + c_{11} N^{1-2/(2^{d+1}-1)}.$$

It follows that

$$A_{d+1}(N) < \beta_{d+1} N^{1-1/(2^{d+1}-1)} + c_{11} N^{1-2/(2^{d+1}-1)}$$

which completes the proof.

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